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The Approximation Functional and Interpolation with Perturbed Continuity

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The continuity conditions at the endpoints of interpolation theorems, $\|Ta\|_{B_j} \leqslant M_j \|a\|_{A_j}$ for $j=0, 1$, can be written with the help of the approximation functional: $\|E(t, Ta; B_1, B_0)\|_{L^\infty} \leqslant M_0 \|a\|_{A_0}$ and $\|E(t, Ta; B_0, B_1)\|_{L^\infty} \leqslant M_1 \|a\|_{A_1}$. As a special case of the results we present here we show that in the hypotheses of the interpolation theorem the L^∞ norms can be replaced by $\mathrm{BMO}(\mathbb{R}_+)$ norms. This leads to a strong version of the Stein-Weiss theorem on interpolation with change of measure. Another application of our results is that the condition $f \in L^0$, i.e., $f_* \in L^\infty$, where $f_*(\gamma) = \mu\{|f| > \gamma\}$ is the distribution function of f, can be replaced in interpolation with $L(p, q)$ spaces by the weaker $f_* \in \mathrm{BMO}(\mathbb{R}_+)$. © 2001 Academic Press

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1. INTRODUCTION

In [10] we proved an interpolation theorem for operators with perturbed continuity at the endpoints of the interpolation scale. This theorem generalizes previous work on weak-type classes in [1, 4, 5, 8, 9]. The perturbations of continuity in [10] were expressed in terms of the K-functional. The K-functional for the interpolation couple (A_1, A_0) is expressed algebraically in terms of the K-functional for interpolation couple (A_0, A_1) and this makes the K-functional a natural tool to develop the general theory.

However, in some very important cases it is much easier to calculate the E-functional than the K-functional. Since the dependence of the interpolation theorem on the estimation of the functionals is rather delicate it seems advantageous to prove an interpolation theorem with the perturbation conditions expressed in terms of the approximation functional. The approximation functional was defined in [6]:

Definition 1.1. Let (A_0, A_1) be an interpolation couple. We define

$$E(t, a; A_0, A_1) = \inf\{\|a - a_0\|_{A_1} \mid \|a_0\|_{A_0} \leqslant t\}.$$

If we take, see [6],

$$\|f\|_{L^0} = \mu\{|f| > 0\}$$

and

$$L^0 = \{f \mid \|f\|_{L^0} < \infty\}$$

then it is easy to see that

$$E(t, f; L^0, L^p) = \left(\int_t^\infty f^*(s)^p \, ds\right)^{1/p}$$

and in particular

$$E(t, f; L^0, L^\infty) = f^*(t).$$

The use of the E-functional enables us to include an interesting class of functions in the interpolation scale:

The condition

$$\frac{1}{t}\int_0^t f^*(s)\, ds - f^*(t) \leqslant C \tag{1.1}$$

introduced in [1] is equivalent to

$$f^* \in \mathrm{BMO}(\mathbb{R}_+). \tag{1.2}$$

The class of functions which satisfy (1.2) replaces L^∞ in the interpolation theorem, giving a stronger result. We will see that the class of functions which satisfy

$$f_* \in \mathrm{BMO}(\mathbb{R}_+),$$

where f_* is the distribution function of f, i.e.,

$$f_*(\gamma) = \mu\{|f| > \gamma\}$$

plays a similar role, replacing the space L^0.

In greater generality, let (A_0, A_1) and (B_0, B_1) be two interpolation couples. Let T be an operator satisfying some quasilinearity conditions. The classical interpolation theorem states that if T satisfies two continuity conditions:

$$\|Ta\|_{B_0} \leqslant M_0 \|a\|_{A_0} \tag{1.3}$$

and

$$\|Ta\|_{B_1} \leqslant M_1 \|a\|_{A_1} \tag{1.4}$$

then

$$\|Ta\|_{(B_0, B_1)_{\theta, q; K}} \leqslant C(\theta, q, M_0, M_1) \|Ta\|_{(A_0, A_1)_{\theta, q; K}}.$$

Conditions (1.3) and (1.4) can be restated in terms of the E-functional for the couple (B_0, B_1). Condition (1.3) is equivalent to

$$\|E(t, Ta; B_0, B_1)\|_{L^0} \leqslant M_0 \|a\|_{A_0}$$

which in turn is equivalent to

$$\|E(t, Ta; B_1, B_0)\|_{L^\infty} \leqslant M_0 \|a\|_{A_0} \tag{1.5}$$

and (1.4) is equivalent to

$$\|E(t, Ta; B_0, B_1)\|_{L^\infty} \leqslant M_1 \|a\|_{A_1}.$$

The main idea of this paper is to replace the L^∞ conditions on E by $\mathrm{BMO}(\mathbb{R}_+)$ conditions. We will get the same interpolation results giving us a stronger interpolation theorems. We will see that for non-increasing functions the BMO($\mathbb{R}_+$) norm is equivalent to

$$\sup_{t>0} (f(t) - f(\gamma t)),$$

where $\gamma > 1$ is arbitrary. Thus the BMO($\mathbb{R}_+$) condition which generalizes (1.5) is equivalent to

$$\sup_{t>0} (E(t, Ta; B_1, B_0) - E(\gamma t, Ta; B_1, B_0)) \leqslant M_0 \|a\|_{A_0}.$$

This can be viewed as a perturbation of the continuity condition (1.5).

We make a further generalization. To state it we need to make the following definition:

DEFINITION 1.2. Let h and g be two non-negative functions on $\mathbb{R}_+$. We write

$$h \overset{E}{\sim} g$$

if there exists $\beta \geqslant 1$ so that for all $t > 0$

$$\frac{1}{\beta} h(\beta t) \leqslant g(t) \leqslant \beta h\left(\frac{t}{\beta}\right).$$

Since the E-functional in some cases is known only up to this equivalence we express the perturbed continuity condition by an arbitrary $h \overset{E}{\sim} E$. The use of the more general function h does not affect the conclusion of the usual interpolation result.

Finally we add two more parameters ε_0 and ε_1 to the perturbation. We leave the statement of their role to the interpolation theorem below.

As an application of the present approach we also get a stronger version of the Stein–Weiss theorem on the interpolation of weighted L^p spaces, see [7].

All interpolation couples in this paper are interpolation couples of quasi-Banach groups. We will assume the standard results of interpolation theory as stated in [2].

2. QUASILINEARITY

The proof of the interpolation theorem will use the corresponding theorem in [10]. This application requires a proof of a theorem which is interesting in its own right: E-quasilinearity implies K-quasilinearity.

It will be useful to make the following definition:

DEFINITION 2.1.

$$K_r(t, b; B_0, B_1) = \inf \{ (\|b_0\|_{B_0}^r + t^r \|b_1\|_{B_1}^r)^{1/r} \mid b = b_0 + b_1 \}. \tag{2.1}$$

We denote $K_1 = K$.[3]

Clearly $K_r \approx K$. Once we know the E-functional we can calculate the K-functional:

$$K_r(t, b; B_1, B_0) = \inf_{s>0} (E^r(s, b; B_0, B_1) + s^r t^r)^{1/r}. \tag{2.2}$$

Interpolation theorems in the K-method and in the E-method are typically proved for operators which are K-quasilinear and E-quasilinear. We recall the definitions:

DEFINITION 2.2. Let (A_0, A_1) and (B_0, B_1) be two interpolation couples. We say that

$$T: A_0 + A_1 \mapsto B_0 + B_1$$

is a K-quasilinear operator if for some $C > 0$, for all $a_j \in A_j$ and all $t > 0$, we have

$$K(t, T(a_0 + a_1); B_0, B_1) \leqslant C(K(t, Ta_0; B_0, B_1) + K(t, Ta_1; B_0, B_1)).$$

DEFINITION 2.3. We say that

$$T: A_0 + A_1 \mapsto B_0 + B_1$$

is a E-quasilinear operator if for some $\beta > 0$, for all $a_j \in A_j$ and all $t > 0$, we have

$$E(t, T(a_0 + a_1); B_0, B_1) \leqslant \beta \left(E\left(\frac{t}{\beta}, Ta_0; B_0, B_1\right) + E\left(\frac{t}{\beta}, Ta_1; B_0, B_1\right)\right).$$

[3] The definition above is somewhat different than that in [2]: instead of the expression on the right-hand side of (2.1), Bergh and Löfstrom take $\inf \{ (\|b_0\|_{B_0}^r + t \|b_1\|_{B_1}^r)^{1/r} \mid b = b_0 + b_1 \}$.

LEMMA 2.4. *If f_1, f_2 are two non-increasing functions on $\mathbb{R}_+$, then*

$$\begin{aligned}&\inf_{s>0}\{\max\{f_1(s), f_2(s), st\}\}\\&\qquad=\max\{\inf_{s>0}\{\max\{f_1(s), st\}\}, \inf_{s>0}\{\max\{f_2(s), st\}\}\}.\end{aligned}$$

Proof. Let

$$s_j=\sup_{s>0}\{f_j(s)\geqslant st\}.$$

It is easy to see that

$$\inf_{s>0}\max\{f_j(s), st\}=s_j t$$

and $f_j(s_j^+)\leqslant s_j t$. Thus if we take $s_0=\max\{s_1, s_2\}$ we have

$$\max\{\inf_{s>0}\{\max\{f_1(s), st\}\}, \inf_{s>0}\{\max\{f_2(s), st\}\}\}=s_0 t$$

and

$$f_j(s_0^+)\leqslant f_j(s_j^+)\leqslant s_j t\leqslant s_0 t$$

so that

$$\begin{aligned}&\inf_{s>0}\{\max\{f_1(s), f_2(s), st\}\}\\&\qquad\leqslant\max\{f_1(s_0^+), f_2(s_0^+), s_0 t\}=s_0 t\\&\qquad=\max\{\inf_{s>0}\{\max\{f_1(s), st\}\}, \inf_{s>0}\{\max\{f_2(s), st\}\}\}.\end{aligned}$$

The opposite inequality holds trivially. ∎

THEOREM 2.5. *If $T: A_0+A_1\mapsto B_0+B_1$ is E-quasilinear, then it is K-quasilinear.*

Proof. From E-quasilinearity it follows that for some $\beta>0$

$$\begin{aligned}E(t, T(a_0+a_1); B_0, B_1)&\leqslant\beta\left(E\left(\frac{t}{\beta}, Ta_0; B_0, B_1\right)+E\left(\frac{t}{\beta}, Ta_1; B_0, B_1\right)\right)\\&\leqslant 2\beta\max\left\{E\left(\frac{t}{\beta}, Ta_0; B_0, B_1\right), E\left(\frac{t}{\beta}, Ta_1; B_0, B_1\right)\right\}.\end{aligned}$$

Let us write $E(\cdot,\ \cdot)=E(\cdot,\ \cdot;B_0,B_1)$. We have

$$
\begin{aligned}
&K(t, T(a_0+a_1); B_1, B_0)\\
&\quad=\inf_{s>0}\,(E(s, T(a_0+a_1))+st)\leqslant 2\inf_{s>0}\max\{E(s, T(a_0+a_1)), st\}\\
&\quad\leqslant 2\inf_{s>0}\max\left\{2\beta\max\left\{E\left(\frac{s}{\beta}, Ta_0\right), E\left(\frac{s}{\beta}, Ta_1\right)\right\}, st\right\}\\
&\quad=4\beta\inf_{s>0}\max\left\{E\left(\frac{s}{\beta}, Ta_0\right), E\left(\frac{s}{\beta}, Ta_1\right), \frac{st}{2\beta}\right\}\\
&\quad=4\beta\inf_{s>0}\max\left\{E(s, Ta_0), E(s, Ta_1), \frac{st}{2}\right\}\\
&\quad\leqslant 4\beta\inf_{s>0}\max\{E(s, Ta_0), E(s, Ta_1), st\}.
\end{aligned}
$$

By Lemma 2.4

$$
\begin{aligned}
&4\beta\inf_{s>0}\max\{E(s, Ta_0), E(s, Ta_1), st\}\\
&\quad=4\beta\max\{\inf_{s>0}\max\{E(s, Ta_0), st\}, \inf_{s>0}\max\{E(s, Ta_1), st\}\}\\
&\quad\leqslant 4\beta\{\inf_{s>0}\{E(s, Ta_0)+st\}+\inf_{s>0}\{E(s, Ta_1)+st\}\}\\
&\quad=4\beta(K(t, Ta_0; B_1, B_0)+K(t, Ta_1; B_1, B_0))
\end{aligned}
$$

so that

$$K(t, T(a_0+a_1); B_1, B_0)\leqslant 4\beta(K(t, Ta_0; B_1, B_0)+K(t, Ta_1; B_1, B_0)). \quad \blacksquare$$

3. INTERPOLATION

Since we will prove the interpolation theorem in the E-method using the corresponding theorem in [10], we quote a version of that theorem here. Throughout, C will denote a generic constant which depends on $\varepsilon_0, \varepsilon_1, r, \theta, \alpha, q, g$ etc.

THEOREM 3.1. *Let* (A_0, A_1) *and* (B_0, B_1) *be two interpolation couples. Let*

$$T\colon A_0+A_1\mapsto B_0+B_1$$

be a K-quasilinear operator. Let

$$g\colon \mathbb{R}_+ \times (A_0 + A_1) \mapsto \mathbb{R}_+$$

be such that

$$g \approx K(\cdot, \cdot; B_0, B_1) \tag{3.1}$$

and that for all $a \in A_0 \cap A_1$

$$g(\gamma t, Ta) \leqslant (M_0^r \|a\|_{A_0}^r + \varepsilon_0^r g^r(t, Ta))^{1/r} \tag{3.2}$$

and

$$\gamma g(t, Ta) \leqslant (M_1^r \|a\|_{A_1}^r\, t^r + \varepsilon_1^r g^r(\gamma t, Ta))^{1/r}, \tag{3.3}$$

where $0 < r < \infty$, $\gamma > 1$, *and* $\varepsilon_0, \varepsilon_1$ *satisfy* $0 \leqslant \varepsilon_0, \varepsilon_1 < \gamma$ *and* $\varepsilon_0 \varepsilon_1 < \gamma$. *Then for all* θ *so that*

$$\frac{\log^+ \varepsilon_0}{\log \gamma} < \theta < 1 - \frac{\log^+ \varepsilon_1}{\log \gamma} \tag{3.4}$$

and $0 < q \leqslant \infty$, *we have*

$$\|Ta\|_{(B_0, B_1)_{\theta, q; K}} \leqslant C M_0^{1-\theta} M_1^{\theta} \|a\|_{(A_0, A_1)_{\theta, q; K}}. \tag{3.5}$$

The terms $\varepsilon_0^r g^r(t, Ta)$ and $\varepsilon_1^r g^r(\gamma t, Ta)$ above express the perturbation of the continuity conditions for T in the standard interpolation theorem which corresponds to the case $\varepsilon_0 = \varepsilon_1 = 0$. We will have similar terms in the E-method. We naturally want to allow the widest possible choice of functions corresponding to g, and so we make the following definition.

Definition 3.2. Given $h\colon \mathbb{R}_+ \mapsto \mathbb{R}_+$ we define

$$\overset{\searrow}{h}(t) = \inf_{s \leqslant t} h(s).$$

Clearly $\overset{\searrow}{h}$ is the greatest non-increasing minorant of h. If h is a non-increasing function, then clearly $\overset{\searrow}{h} = h$.

Theorem 3.3. *Let* (A_0, A_1) *and* (B_0, B_1) *be two interpolation couples. Let*

$$T\colon A_0 + A_1 \mapsto B_0 + B_1$$

be an E-quasilinear operator. Assume that

$$h\colon \mathbb{R}_+ \times (B_0 + B_1) \mapsto \mathbb{R}_+$$

satisfies

$$\vec{h} \overset{E}{\sim} E(\cdot,\ \cdot;B_1, B_0) \tag{3.6}$$

and that for all $a \in A_0 \cap A_1$ *and* $t > 0$

$$h\left(\frac{t}{\gamma}, Ta\right) \leqslant \left(M_0^r\,\|a\|_{A_0}^r + \varepsilon_0^r h^r\left(\frac{t}{\varepsilon_0}, Ta\right)\right)^{1/r} \tag{3.7}$$

and

$$h((t^r + M_1^r\,\|a\|_{A_1}^r)^{1/r}, Ta) \leqslant \frac{\varepsilon_1}{\gamma}\, h\left(\frac{t}{\varepsilon_1}, Ta\right), \tag{3.8}$$

where $0 < r < \infty$, $\gamma > 1$, *and* $\varepsilon_0, \varepsilon_1$ *satisfy* $0 \leqslant \varepsilon_0, \varepsilon_1 < \gamma$ *and* $\varepsilon_0\varepsilon_1 < \gamma$. *Then for all* α *so that*

$$\frac{\log^+ \varepsilon_1}{\log \gamma - \log^+ \varepsilon_1} < \alpha < \frac{\log \gamma - \log^+ \varepsilon_0}{\log^+ \varepsilon_0} \tag{3.9}$$

and[4] *all* $0 < p \leqslant \infty$, *we have*

$$\|Ta\|_{(B_0, B_1)_{\alpha, p;E}} \leqslant C M_0^{\alpha} M_1\,\|a\|_{(A_0, A_1)_{\alpha, p;E}}. \tag{3.10}$$

Conditions (3.7) and (3.8) merit some discussion. If $\varepsilon_0 = 0$ we interpret

$$\varepsilon_0^r h^r\left(\frac{t}{\varepsilon_0}, Ta\right) = 0$$

and in this case (3.7) reads

$$h\left(\frac{t}{\gamma}, Ta\right) \leqslant M_0\,\|a\|_{A_0},$$

i.e.,

$$\|h(\cdot, Ta)\|_{L^\infty} \leqslant M_0\,\|a\|_{A_0}.$$

[4] Observe that if $\varepsilon_0, \varepsilon_1 \leqslant 1$, then $0 < \alpha < \infty$.

By (3.6) this implies

$$\|E(\cdot, Ta; B_1, B_0)\|_{L^\infty} \leqslant \beta M_0 \|a\|_{A_0}.$$

Thus the term $\varepsilon_0^r h^r(t/\varepsilon_0, Ta)$ is a perturbation of the usual continuity condition.

If $\varepsilon_1 = 0$ condition (3.8) reads

$$h((t^r + M_1^r \|a\|_{A_1}^r)^{1/r}, Ta) = 0$$

for all $t > 0$. This implies

$$\|h(\cdot, Ta)\|_{L^0} \leqslant M_1 \|a\|_{A_1}$$

so that

$$\|E(\cdot, Ta; B_1, B_0)\|_{L^0} \leqslant \beta M_1 \|a\|_{A_1}$$

which is equivalent to

$$\|E(\cdot, Ta; B_0, B_1)\|_{L^\infty} \leqslant \beta M_1 \|a\|_{A_1}.$$

This is, of course, the usual continuity condition.

Proof. We define for $b \in B_0 + B_1$

$$g(t, b) = \inf_{s>0} \left\{ \left[\overset{\searrow}{h}(s, b) \right]^r + t^r s^r \right\}^{1/r}. \tag{3.11}$$

From (3.6)

$$\frac{1}{\beta} g(t, b) = \frac{1}{\beta} \inf_{s>0} ([\overset{\searrow}{h}(\beta s, b)]^r + t^r \beta^r s^r)^{1/r} \leqslant \inf_{s>0} (E^r(s, b; B_1, B_0) + t^r s^r)^{1/r}$$

$$\leqslant \inf_{s>0} \left(\left[\beta \overset{\searrow}{h} \left(\frac{s}{\beta}, b \right) \right]^r + \beta^r t^r \beta^{-r} s^r \right)^{1/r} = \inf_{s>0} ([\beta \overset{\searrow}{h}(s, b)]^r + \beta^r t^r s^r)^{1/r}$$

$$= \beta g(t, b)$$

so that by (2.2)

$$\frac{1}{\beta} g(t, b) \leqslant K_r(t, b; B_0, B_1) \leqslant \beta g(t, b). \tag{3.12}$$

Of course,

$$K_r(t, b; B_0, B_1) \approx K(t, b; B_0, B_1)$$

so that we have (3.1).

From (3.7) follows

$$\overset{\searrow}{h}\left(\frac{t}{\gamma}, Ta\right) \leqslant \left[\left(M_0^r \|a\|_{A_0}^r + \varepsilon_0^r h^r\left(\frac{s}{\varepsilon_0}, Ta\right)\right)^{1/r}\right]^{\searrow}(t)$$

$$= \left(M_0^r \|a\|_{A_0}^r + \varepsilon_0^r (\overset{\searrow}{h})^r\left(\frac{t}{\varepsilon_0}, Ta\right)\right)^{1/r}$$

so that

$$g^r(\gamma t, Ta) = \inf_{s>0} \{[\overset{\searrow}{h}(s, Ta)]^r + \gamma^r t^r s^r\}$$

$$= \inf_{s>0} \left\{\left[\overset{\searrow}{h}\left(\frac{s}{\gamma}, Ta\right)\right]^r + t^r s^r\right\}$$

$$\leqslant \inf_{s>0} \left(M_0^r \|a\|_{A_0}^r + \left[\varepsilon_0 \overset{\searrow}{h}\left(\frac{s}{\varepsilon_0}, Ta\right)\right]^r + t^r s^r\right)$$

$$= M_0^r \|a\|_{A_0}^r + \varepsilon_0^r \inf_{s>0} \left(\left[\overset{\searrow}{h}\left(\frac{s}{\varepsilon_0}, Ta\right)\right]^r + t^r \left(\frac{s}{\varepsilon_0}\right)^r\right)$$

$$= M_0^r \|a\|_{A_0}^r + \varepsilon_0^r \inf_{s>0} ([\overset{\searrow}{h}(s, Ta)]^r + t^r s^r)$$

$$= M_0^r \|a\|_{A_0}^r + \varepsilon_0^r g^r(t, Ta)$$

proving (3.2).

From (3.8) follows

$$[h((s^r + M_1^r \|a\|_{A_1}^r)^{1/r}, Ta)]^{\searrow}(t) \leqslant \frac{\varepsilon_1}{\gamma} \overset{\searrow}{h}\left(\frac{t}{\varepsilon_1}, Ta\right).$$

But

$$[h((s^r + M_1^r \|a\|_{A_1}^r)^{1/r}, Ta)]^{\searrow}(t)$$

$$= \inf_{0<s\leqslant t} h((s^r + M_1^r \|a\|_{A_1}^r)^{1/r}, Ta)$$

$$= \inf\{h(s, Ta) \mid M_1 \|a\|_{A_1} < s \leqslant (t^r + M_1^r \|a\|_{A_1}^r)^{1/r}\}$$

$$\geqslant \inf\{h(s, Ta) \mid 0 < s \leqslant (t^r + M_1^r \|a\|_{A_1}^r)^{1/r}\} = \overset{\searrow}{h}((t^r + M_1^r \|a\|_{A_1}^r)^{1/r}, Ta)$$

so that

$$\vec{h}((t^r+M_1^r\,\|a\|_{A_1}^r)^{1/r},\,Ta)\leqslant\frac{\varepsilon_1}{\gamma}\vec{h}\left(\frac{t}{\varepsilon_1},\,Ta\right).$$

Thus

$$\begin{aligned}
\gamma^r g^r(t,\,Ta) &= \gamma^r \inf_{s>0}\{[\vec{h}(s,\,Ta)]^r+t^r s^r\}\\
&\leqslant \gamma^r \inf_{s>M_1\,\|a\|_{A_1}}\{[\vec{h}(s,\,Ta)]^r+t^r s^r\}\\
&= \gamma^r \inf_{u>0}\{[\vec{h}((u^r+M_1^r\,\|a\|_{A_1}^r)^{1/r},\,Ta)]^r+t^r(u^r+M_1^r\,\|a\|_{A_1}^r)\}\\
&\leqslant \gamma^r \inf_{u>0}\left\{\left[\frac{\varepsilon_1}{\gamma}\,\vec{h}\left(\frac{u}{\varepsilon_1},\,Ta\right)\right]^r+t^r(u^r+M_1^r\,\|a\|_{A_1}^r)\right\}\\
&= \gamma^r M_1^r\,\|a\|_{A_1}^r\,t^r+\gamma^r \inf_{u>0}\left\{\left[\frac{\varepsilon_1}{\gamma}\,\vec{h}\left(\frac{u}{\varepsilon_1},\,Ta\right)\right]^r+t^r u^r\right\}\\
&= \gamma^r M_1^r\,\|a\|_{A_1}^r\,t^r+\varepsilon_1^r \inf_{u>0}\left\{\left[\vec{h}\left(\frac{u}{\varepsilon_1},\,Ta\right)\right]^r+\gamma^r t^r\left(\frac{u}{\varepsilon_1}\right)^r\right\}\\
&= \gamma^r M_1^r\,\|a\|_{A_1}^r\,t^r+\varepsilon_1^r g^r(\gamma t)
\end{aligned}$$

proving (3.3). All conditions of Theorem 3.1 are met and therefore (3.5) holds. We recall

$$\|a\|_{(A_0,\,A_1)_{\theta,\,q;\,K}}^{1/\theta}\approx\|a\|_{(A_0,\,A_1)_{\alpha,\,p;\,E}},\tag{3.13}$$

where $\theta=\frac{1}{\alpha+1}$ and $p=\theta q$ (see [2, Theorem 7.1.7]) and get (3.10). ∎

Definition 3.4. Let (A_0, A_1) be an interpolation couple and let $0\leqslant\varepsilon<\gamma$, $1<\gamma$, and $0<r<\infty$. Let

$$h\colon \mathbb{R}_+\times(A_0+A_1)\mapsto\mathbb{R}_+$$

be such that $\vec{h}\overset{E}{\sim}E(\cdot,\,\cdot;A_1,A_0)$. We define

$$\|a\|_{W_E(A_0,\,A_1;\,\varepsilon,\,\gamma,\,r,\,h)}=\sup_{t>0}\left[h^r\left(\frac{t}{\gamma},\,a\right)-\varepsilon^r h^r\left(\frac{t}{\varepsilon},\,a\right)\right]_+^{1/r}$$

and

$$W_E(A_0, A_1; \varepsilon, \gamma, r, h) = \{a \in A_0 + A_1 \mid \|a\|_{W_E(A_0, A_1; \varepsilon, \gamma, r, h)} < \infty\}.$$

We denote $W_E(A_0, A_1; \varepsilon, \gamma, r, E(\cdot, \cdot; A_1, A_0))$ by $W_E(A_0, A_1; \varepsilon, \gamma, r)$.

With this definition condition (3.7) can be written

$$\|Ta\|_{W_E(B_0, B_1; \varepsilon, \gamma, r, h)} \leqslant M_0 \|a\|_{A_0}.$$

The E-functional for the couple (A_1, A_0) is controlled by the generalized inverse of the E-functional for the couple (A_0, A_1). This enables us in some case to express (3.8) by a norm inequality for an appropriate W_E class.

DEFINITION 3.5. Let $h: \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a non-increasing function. We define

$$h^{-1}(t) = \inf\{s > 0 \mid h(s) \leqslant t\}.$$

It is easy to see that h^{-1} is a right-continuous non-increasing function and that for all $s, t \in R^+$

$$h^{-1}(h(s)) \leqslant s \qquad \text{and} \qquad h([h^{-1}(t)]^+) \leqslant t. \tag{3.14}$$

THEOREM 3.6.

$$E(t^+, a; A_0, A_1) \leqslant E^{-1}(t, a; A_1, A_0) \leqslant E(t, a; A_0, A_1) \tag{3.15}$$

Proof. Let us consider the first inequality. For all s so that $E(s, a; A_1, A_0) \leqslant t$ and all $\varepsilon > 0$ we have $a_1 \in A_1$ so that $\|a_1\| \leqslant s$ and so that $\|a - a_1\|_{A_0} \leqslant t + \varepsilon$. This implies $E(t + \varepsilon, a; A_0, A_1) \leqslant s$ and so for all $\varepsilon > 0$

$$E(t + \varepsilon, a; A_0, A_1) \leqslant \inf\{s > 0 \mid E(s, a; A_1, A_0) \leqslant t\} = E^{-1}(t, a; A_1, A_0)$$

which implies

$$E(t^+, a; A_0, A_1) \leqslant E^{-1}(t, a; A_1, A_0).$$

Let us consider the second inequality. For all $\varepsilon > 0$ there exists $a_0 \in A_0$ so that $\|a_0\|_{A_0} \leqslant t$ and so that

$$\|a - a_0\|_{A_1} \leqslant E(t, a; A_0, A_1) + \varepsilon.$$

Thus

$$E(E(t, a; A_0, A_1) + \varepsilon, a; A_1, A_0) \leqslant t$$

and therefore, for all $\varepsilon > 0$

$$\inf\{s > 0 \mid E(s, a; A_1, A_0) \leqslant t\} \leqslant E(t, a; A_0, A_1) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$E^{-1}(t, a; A_1, A_0) = \inf\{s > 0 \mid E(s, a; A_1, A_0) \leqslant t\} \leqslant E(t, a; A_0, A_1). \quad \blacksquare$$

If we assume in Theorem 3.6 that the function h is non-increasing we get a symmetric version of the theorem. This version is the most convenient one for the applications which we present in the last section.

THEOREM 3.7. *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples. Let*

$$T: A_0 + A_1 \mapsto B_0 + B_1$$

be an E-quasilinear operator. Assume that

$$h: \mathbb{R}_+ \times (B_0 + B_1) \mapsto \mathbb{R}_+$$

satisfies $h \overset{E}{\sim} E(\cdot, \cdot; B_1, B_0)$ and $h(\cdot, b)$ is a non-increasing function. Assume also that for all $a \in A_0 \cap A_1$

$$\|Ta\|_{W_E(B_0, B_1; \varepsilon_0, \gamma, r, h)} \leqslant M_0 \|a\|_{A_0} \tag{3.16}$$

and

$$\|Ta\|_{W_E(B_1, B_0; \varepsilon_1, \gamma, r, h^{-1})} \leqslant M_1 \|a\|_{A_1} \tag{3.17}$$

where $0 < r < \infty$, $\gamma > 1$, and $\varepsilon_0, \varepsilon_1$ satisfy $0 \leqslant \varepsilon_0, \varepsilon_1 < \gamma$ and $\varepsilon_0 \varepsilon_1 < \gamma$. Then for all α so that

$$\frac{\log^+ \varepsilon_1}{\log \gamma - \log^+ \varepsilon_1} < \alpha < \frac{\log \gamma - \log^+ \varepsilon_0}{\log^+ \varepsilon_0} \tag{3.18}$$

and all $0 < q \leqslant \infty$, we have

$$\|Ta\|_{(B_0, B_1)_{\alpha, q; E}} \leqslant C M_0^\alpha M_1 \|a\|_{(A_0, A_1)_{\alpha, q; E}}. \tag{3.19}$$

Proof. We need to show that if $h: \mathbb{R}_+ \times (A_0 + A_1) \mapsto \mathbb{R}_+$ defines a W_E class, i.e., if

$$h \overset{E}{\sim} E(\cdot, \cdot; B_1, B_0) \tag{3.20}$$

and for all $a \in A_0 + A_1$ the function $h(\cdot, a)$ is non-increasing, then

$$h^{-1} \overset{E}{\sim} E(\cdot, \cdot; B_0, B_1) \tag{3.21}$$

and so h^{-1} can be used to define W_E classes.

From (3.20) there exists $\beta \geqslant 1$ so that

$$\left[\frac{1}{\beta} h(\beta s, a)\right]^{-1}(t) \leqslant E^{-1}(t, a; A_1, A_0) \leqslant \left[\beta h\left(\frac{s}{\beta}, a\right)\right]^{-1}(t).$$

Clearly

$$\left[\frac{1}{\beta} h(\beta s, a)\right]^{-1}(t) = \frac{1}{\beta} h^{-1}(\beta t, a)$$

so that

$$\frac{1}{\beta} h^{-1}(\beta t, a) \leqslant E^{-1}(t, a; A_1, A_0) \leqslant \beta h^{-1}\left(\frac{t}{\beta}, a\right).$$

By (3.15)

$$\frac{1}{\beta} h^{-1}(\beta t, a) \leqslant E(t, a; A_0, A_1)$$

and

$$E(2t, a; A_0, A_1) \leqslant E(t^+, a; A_0, A_1) \leqslant E^{-1}(t, a; A_1, A_0) \leqslant \beta h^{-1}\left(\frac{t}{\beta}, a\right)$$

so that (3.21) holds.

We define the function g as in (3.11), noting that $h = \overset{\searrow}{h}$

$$g(t) = \inf_{s>0} \{h^r(s) + t^r s^r\}^{1/r}.$$

As in the proof of Theorem 3.3 we get both (3.1) and (3.2). Let us see that (3.3) holds. Condition (3.17) is equivalent to

$$h^{-1}\left(\frac{t}{\gamma}, Ta\right) \leqslant \left(M_1^r \|a\|_{A_1}^r + \left[\varepsilon_1 h^{-1}\left(\frac{t}{\varepsilon_1}, Ta\right)\right]^r\right)^{1/r}. \tag{3.22}$$

But

$$g^r(t) = \inf_{u>0} \{h^r(u, Ta) + t^r u^r\} = \inf_{u>0} \left(\inf_{s \geqslant h(u, Ta)} \{s^r + t^r u^r\} \right)$$

$$= \inf_{s>0} \left(\inf_{\{u \mid h(u, Ta) \leqslant s\}} \{s^r + t^r u^r\} \right) = \inf_{s>0} \{s^r + [th^{-1}(s, Ta)]^r\}.$$

From (3.22) we have

$$\inf_{s>0} \left\{ \left[th^{-1}\left(\frac{s}{\gamma}, Ta\right)\right]^r + s^r \right\}$$

$$\leqslant \inf_{s>0} \left\{ M_1^r t^r \|a\|_{A_1}^r + \left[t\varepsilon_1 h^{-1}\left(\frac{s}{\varepsilon_1}, Ta\right)\right]^r + s^r \right\}$$

$$= M_1^r t^r \|a\|_{A_1}^r + \varepsilon_1^r \inf_{s>0} \left\{ \left[th^{-1}\left(\frac{s}{\varepsilon_1}, Ta\right)\right]^r + \left(\frac{s}{\varepsilon_1}\right)^r \right\}$$

$$= M_1^r t^r \|a\|_{A_1}^r + \varepsilon_1^r g^r(t).$$

On the other hand

$$\inf_{s>0} \left\{ \left[th^{-1}\left(\frac{s}{\gamma}, Ta\right)\right]^r + s^r \right\} = \gamma^r \inf_{s>0} \left\{ \left[\frac{t}{\gamma} h^{-1}\left(\frac{s}{\gamma}, Ta\right)\right]^r + \left(\frac{s}{\gamma}\right)^r \right\}$$

$$= \gamma^r g^r\left(\frac{t}{\gamma}\right)$$

so that

$$\gamma^r g^r\left(\frac{t}{\gamma}\right) \leqslant M_1^r t^r \|a\|_{A_1}^r + \varepsilon_1^r g^r(t).$$

As in the conclusion of Theorem 3.3 all conditions of Theorem 3.1 are met and therefore (3.5) and so also (3.19) hold. ∎

We will prove a reiteration theorem for W_E classes. The proof will use the corresponding theorem in [10]; we quote that theorem here.

THEOREM 3.8. *Let (A_0, A_1) and (B_0, B_1) be two interpolation couples. Let*

$$T: A_0 + A_1 \mapsto B_0 + (B_0, B_1)_{\theta, q; K}$$

be a K-quasilinear operator and let $g \approx K(\cdot, \cdot; B_0, B_1)$, $0 < q < \infty$, $0 \leqslant \varepsilon < \gamma$, *and* $1 < \gamma$. *Assume that for all* $a \in A_0 \cap A_1$:

$$\sup_{t>0} \left[g^q(\gamma t, Ta) - \varepsilon^q g^q(t, Ta) \right]^{1/q} \leqslant M_0 \, \|a\|_{A_0} \tag{3.23}$$

and

$$\|Ta\|_{(B_0, B_1)_{\theta, q; K}} \leqslant M_1 \, \|a\|_{A_1}, \tag{3.24}$$

where

$$\frac{\log^+ \varepsilon}{\log \gamma} < \theta < 1. \tag{3.25}$$

Then for

$$\frac{\log^+ \varepsilon}{\theta \log \gamma} < \lambda < 1 \tag{3.26}$$

we have for all $0 < q_1 \leqslant \infty$,

$$\|Ta\|_{(B_0, B_1)_{\lambda\theta, q_1; K}} \leqslant C M_0^{1-\lambda} M_1^{\lambda} \, \|a\|_{(A_0, A_1)_{\lambda, q_1; K}}. \tag{3.27}$$

THEOREM 3.9 (Reiteration Theorem). *Let* (A_0, A_1) *and* (B_0, B_1) *be two interpolation couples. Let*

$$T: A_0 + A_1 \mapsto B_0 + (B_1, B_0)_{s, p: E}$$

be a E-quasilinear operator which for all $a \in A_0 \cap A_1$ *satisfies*

$$\|Ta\|_{W_E(B_0, B_1; \varepsilon, \gamma, p, h)} \leqslant M_0 \, \|a\|_{A_0} \tag{3.28}$$

and

$$\|Ta\|_{(B_1, B_0)_{s, p; E}} \leqslant M_1 \, \|a\|_{A_1}, \tag{3.29}$$

where $0 < p < \infty$, $0 \leqslant \varepsilon < \gamma$, $\gamma > 1$ *and*

$$\frac{\log^+ \varepsilon}{\log \gamma - \log^+ \varepsilon} < s. \tag{3.30}$$

Then for

$$\frac{1}{s} \cdot \frac{\log^+ \varepsilon}{\log \gamma - \log^+ \varepsilon} < \eta < 1 \tag{3.31}$$

we have for all $0 < q \leqslant \infty$,

$$\|Ta\|_{(B_1, B_0)_{\eta s, q; E}} \leqslant CM_0^{1-\eta} M_1^{\eta} \|a\|_{(A_0, A_1)_{\eta, q; K}}. \tag{3.32}$$

Proof. We will not keep track of the constants which appear in the proof. We define $g(t, a)$ as in (3.11). As in the proof of Theorem 3.3 we have $g(t, b) \approx K(t, b; B_0, B_1)$ and, using (3.28) we also have (3.23). By (3.13) condition (3.29) is equivalent to

$$\|Ta\|_{(B_0, B_1)_{s/(1+s), p(1+s); K}} = \|Ta\|_{(B_1, B_0)_{1/(1+s), p(1+s); K}} \leqslant CM_1^{1/(1+s)} \|a\|_{A_1}^{1/(1+s)}$$

giving us (3.24) with $CM_1^{1/(1+s)}$ in place of M_1. Therefore the hypotheses of Theorem 3.8 hold with $A_1^{1/(1+s)}$ in place of A_1. Condition (3.30) with

$$\theta = \frac{s}{1+s}$$

is equivalent to (3.25). We define λ by

$$\lambda = \eta \, \frac{1+s}{1+s\eta}.$$

Condition (3.31) is equivalent to (3.26) and so, by (3.27), for all $0 < q_1 \leqslant \infty$ we have

$$\begin{aligned} \|Ta\|_{(B_1, B_0)_{(1+s(1-\lambda))/(1+s), q_1; K}} &= \|Ta\|_{(B_0, B_1)_{\lambda s/(1+s), q_1; K}} \\ &\leqslant CM_0^{1-\lambda} M_1^{\lambda/(1+s)} \|a\|_{(A_0, A_1^{1/(1+s)})_{\lambda, q_1; K}}. \end{aligned}$$

We use (3.13) again and get

$$\|Ta\|^{(1+s(1-\lambda))/(1+s)}_{(B_1, B_0)_{s\lambda/(1+s(1-\lambda)), (1+s(1-\lambda))/(1+s) q_1; E}} \leqslant CM_0^{1-\lambda} M_1^{\lambda/(1+s)} \|a\|_{(A_0, A_1^{1/(1+s)})_{\lambda, q_1; K}}.$$

Also, using the Power Theorem, see Theorem 3.11.6 in [2], we have

$$(A_0, A_1^{1/(1+s)})_{\lambda, q_1; K} = (A_0, A_1)^{(1+s(1-\lambda))/(1+s)}_{\lambda/(1+s(1-\lambda)), (1+s(1-\lambda))/(1+s) q_1; K}$$

so that

$$\begin{aligned} &\|Ta\|^{(1+s(1-\lambda))/(1+s)}_{(B_1, B_0)_{s\lambda/(1+s(1-\lambda)), (1+s(1-\lambda))/(1+s) q_1; E}} \\ &\quad \leqslant CM_0^{1-\lambda} M_1^{\lambda/(1+s)} \|a\|^{(1+s(1-\lambda))/(1+s)}_{(A_0, A_1)_{\lambda/(1+s(1-\lambda)), (1+s(1-\lambda)/(1+s) q_1; K}}. \end{aligned}$$

We have

$$\eta = \frac{\lambda}{1 + s(1-\lambda)}$$

and denote

$$q = \frac{1 + s(1-\lambda)}{1+s}\, q_1 .$$

We finally get (3.32). ∎

4. W_E AND NON-INCREASING BMO($\mathbb{R}_+$) FUNCTIONS

Let us see the connection between W_E and BMO($\mathbb{R}_+$) conditions on the E-functional. We will denote by D the cone of non-negative and non-increasing functions on $\mathbb{R}_+$. We recall

THEOREM 4.1 [10]. *If $f \in D \cap L^1_{loc}$, then*

$$\frac{1}{2}\, \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} \leqslant \sup_{t>0} \left(\frac{1}{t} \int_0^t f(u)\, du - f(t) \right) \leqslant 8\, \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} .$$

For $\gamma > 1$

$$\begin{aligned} \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} &\leqslant \sup_{t>0} \left(\frac{1}{t} \int_0^t f(u)\, du - \frac{1}{(\gamma-1)\, t} \int_t^{\gamma t} f(u)\, du \right) \\ &\leqslant \frac{2\gamma^2}{\gamma-1}\, \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} . \end{aligned}$$

Using this theorem we prove a third equivalent expression for $\|f\|_{\mathbf{BMO}(\mathbb{R}_+)}$.

THEOREM 4.2. *If $f \in D \cap L^1_{loc}$ and $\gamma > 1$, then*

$$\frac{\gamma-1}{\gamma}\, \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} \leqslant \sup_{t>0} \left(f(t) - f(\gamma t) \right) \leqslant \left(\frac{2\gamma^2}{\gamma-1} + 8 \right) \|f\|_{\mathbf{BMO}(\mathbb{R}_+)} .$$

Proof.

$$\begin{aligned}\sup_{t>0}(f(t)-f(\gamma t)) &\geqslant \frac{1}{t}\int_0^t (f(u)-f(\gamma u))\,du\\ &=\frac{1}{t}\int_0^t f(u)\,du-\frac{1}{\gamma t}\int_0^{\gamma t} f(u)\,du\\ &=\frac{\gamma-1}{\gamma}\left(\frac{1}{t}\int_0^t f(u)\,du-\frac{1}{(\gamma-1)\,t}\int_t^{\gamma t} f(u)\,du\right)\\ &\geqslant\frac{\gamma-1}{\gamma}\,\|f\|_{\mathrm{BMO}(\mathbb{R}_+)}.\end{aligned}$$

We also have

$$\begin{aligned}f(t)-f(\gamma t) &\leqslant \frac{1}{t}\int_0^t f(u)\,du-f(\gamma t)\\ &=\left(\frac{1}{t}\int_0^t f(u)\,du-\frac{1}{(\gamma-1)\,t}\int_t^{\gamma t} f(u)\,du\right)\\ &\quad+\left(\frac{1}{(\gamma-1)\,t}\int_t^{\gamma t} f(u)\,du-f(\gamma t)\right)\\ &\leqslant\frac{2\gamma^2}{\gamma-1}\,\|f\|_{\mathrm{BMO}(\mathbb{R}_+)}+\left(\frac{1}{\gamma t}\int_0^{\gamma t} f(u)\,du-f(\gamma t)\right)\\ &\leqslant\frac{2\gamma^2}{\gamma-1}\,\|f\|_{\mathrm{BMO}(\mathbb{R}_+)}+8\,\|f\|_{\mathrm{BMO}(\mathbb{R}_+)}.\quad\blacksquare\end{aligned}$$

We therefore have that if for $f\in D\cap L^1_{loc}$

$$\sup_{t>0}\,(f(t)-f(\gamma t))<\infty$$

for one $\gamma>1$, then the inequality holds for all $\gamma>1$.

Recall the definition of $W_E(A_0, A_1; \varepsilon, \gamma, r)$:

$$\|a\|_{W_E(A_0,\,A_1;\,\varepsilon,\,\gamma,\,r)}=\sup_{t>0}\left[\,E^r\left(\frac{t}{\gamma},a;A_0,A_1\right)-\varepsilon^r E^r\left(\frac{t}{\varepsilon},a;A_0,A_1\right)\right]_+^{1/r}.$$

Thus if $\varepsilon=1$ we have

$$\|a\|_{W_E(A_0,\,A_1;\,1,\,\gamma,\,r)}\approx\|E^r(\,\cdot,a;A_0,A_1)\|^{1/r}_{\mathrm{BMO}(\mathbb{R}_+)}$$

and by Theorem 4.2 we have that if $1 < \gamma_1, \gamma_2$, then $W_E(A_0, A_1; 1, \gamma_1, r) = W_E(A_0, A_1; 1, \gamma_2, r)$.

5. APPLICATIONS

Let us calculate

$$\|f\|_{W_E(L^0, L^\infty; \varepsilon, \gamma, 1)} = \sup_{t>0} \left(E\left(\frac{t}{\gamma}, f; L^\infty, L^0\right) - \varepsilon E\left(\frac{t}{\varepsilon}, f; L^\infty, L^0\right)\right)_+$$

$$= \sup_{t>0} \left(f_*\left(\frac{t}{\gamma}\right) - \varepsilon f_*\left(\frac{t}{\varepsilon}\right)\right)_+ .$$

We have shown that if $f \in D \cap L^1_{loc}$, then

$$f \in \mathrm{BMO}(\mathbb{R}_+) \Leftrightarrow \sup_{t>0} \left(f\left(\frac{t}{2}\right) - f(t)\right) < \infty.$$

Thus

$$f \in W_E(L^0, L^\infty; 1, 2, 1) \Leftrightarrow f_* \in \mathrm{BMO}(\mathbb{R}_+).$$

We recall $f^* = (f_*)^{-1}$ is the non-increasing rearrangement of f, i.e., $(f^*)_* = f_*$. Using Theorem 3.7 also with $h = f_*$ we get:

THEOREM 5.1. *Let (A_0, A_1) be an interpolation couple and let $T: A_0 + A_1 \mapsto L^0 + L^\infty$ be an E-quasilinear operator which satisfies for all $a \in A_0 \cap A_1$*

$$\|(Ta)_*\|_{\mathrm{BMO}(\mathbb{R}_+)} \leqslant M_0 \|a\|_{A_0}$$

and

$$\|(Ta)^*\|_{\mathrm{BMO}(\mathbb{R}_+)} \leqslant M_1 \|a\|_{A_1}.$$

Then for all $0 < \alpha < \infty$ and $0 < q \leqslant \infty$

$$\|Ta\|_{(L^0, L^\infty)_{\alpha, q; E}} \leqslant C(\alpha)\, M_0^\alpha M_1 \|a\|_{(A_0, A_1)_{\alpha, q; E}}.$$

In particular, for all $0 < p < \infty$ and $0 < q \leqslant \infty$

$$f^* \in \mathrm{BMO}(\mathbb{R}_+) \quad \text{and} \quad f_* \in \mathrm{BMO}(\mathbb{R}_+) \Rightarrow f \in L(p, q).$$

We consider a second application. Let ω be a positive function measurable on a measure space. We denote

$$\|f\|_{L^p_\omega} = \left(\int |\omega f|^p \right)^{1/p} d\mu.$$

Stein and Weiss in [7] proved in effect,

$$(L^p, L^p_\omega)_{\theta, p; K} = L^p_{\omega^\theta}.$$

Thus if (A_0, A_1) is an interpolation couple and if T is a quasilinear operator into the space of measurable functions on a measure space (Ω, Σ, μ) which satisfies

$$\left(\int_\Omega |Ta|^p \, d\mu \right)^{1/p} \leqslant M_0 \, \|a\|_{A_0} \tag{5.1}$$

and

$$\left(\int_\Omega |\omega Ta|^p \, d\mu \right)^{1/p} \leqslant M_1 \, \|a\|_{A_1} \tag{5.2}$$

then for $0 < \theta < 1$

$$\left(\int_\Omega |\omega^\theta Ta|^p \, d\mu \right)^{1/p} \leqslant C(\theta) \, M_0^{1-\theta} M_1^\theta \, \|a\|_{(A_0, A_1)_{\theta, p; K}}. \tag{5.3}$$

We will prove that (5.1) can be replaced by the weaker condition

$$\sup_{-\infty < n < \infty} \left(\int_{\{\gamma^{n-1} < \omega \leqslant \gamma^n\}} |Ta|^p \, d\mu \right)^{1/p} \leqslant M_0 \, \|a\|_{A_0}, \tag{5.4}$$

where $\gamma > 1$, i.e., this condition, together with (5.2), implies (5.3).

Peetre and Sparr in [6] defined $\|f\|_{\Lambda_\omega} = \|\omega I_{\{f \neq 0\}}\|_{L^\infty}$ and, of course, $\Lambda_\omega = \{f \mid \|f\|_{\Lambda_\omega} < \infty\}$. Λ_ω is a normed group, and is, in fact, complete. If A is any quasi-Banach lattice with a group structure, then it is easy to see that

$$E(t, f; \Lambda_\omega, A) = \|f I_{\{\omega > t\}}\|_A.$$

In particular

$$E(t, f; \Lambda_\omega, L^p) = \|f I_{\{\omega > t\}}\|_{L^p} = \left(\int_{\{\omega > t\}} |f|^p \, d\mu \right)^{1/p}. \tag{5.5}$$

This can then be used to prove that for $0<s<1$ and $0<p\leqslant\infty$

$$(\Lambda_\omega, L^p)_{s,\,p;\,E}=L^p_{\omega^s}. \tag{5.6}$$

The calculation of the E-functional enables us to calculate the norm in $W_E(L^p, \Lambda_\omega; 1, \gamma, p)$ for $1<\gamma<\infty$ and $0<p<\infty$,

$$\begin{aligned}\|f\|_{W_E(L^p,\,\Lambda_\omega;\,1,\,\gamma,\,p)} &= \sup_{t>0}\left(E^p\left(\frac{t}{\gamma}, f; \Lambda_\omega, L^p\right)-E^p(t, f; \Lambda_\omega, L^p)\right)_{1/p}\\ &= \sup_{t>0}\left[\int_{\{\omega>t/\gamma\}}|f|^p\,d\mu-\int_{\{\omega>t\}}|f|^p\,d\mu\right]^{1/p}\\ &= \sup_{t>0}\left[\int_{\{t/\gamma<\omega\leqslant t\}}|f|^p\,d\mu\right]^{1/p}.\end{aligned}$$

It follows that for all $\alpha>0$

$$\|f\|_{W_E(L^p,\,\Lambda_{\omega^\alpha};\,1,\,\gamma^\alpha,\,p)}=\|f\|_{W_E(L^p,\,\Lambda_\omega;\,1,\,\gamma,\,p)}.$$

It also follows that

$$\begin{aligned}\sup_{-\infty<n<\infty}\left[\int_{\{\gamma^n<\omega\leqslant\gamma^{n+1}\}}|f|^p\,d\mu\right]^{1/p} &\leqslant \|f\|_{W_E(L^p,\,\Lambda_\omega;\,1,\,\gamma,\,p)}\\ &\leqslant \gamma\sup_{-\infty<n<\infty}\left[\int_{\{\gamma^n<\omega\leqslant\gamma^{n+1}\}}|f|^p\,d\mu\right]^{1/p}.\end{aligned} \tag{5.7}$$

Recall that $a\in W_E(A_0, A_1; 1, \gamma, p)\Leftrightarrow E^p(\cdot, a; A_0, A_1)\in \mathrm{BMO}(\mathbb{R}_+)$ and that in this case the class $W_E(A_0, A_1; 1, \gamma, p)$ does not depend on the value of $\gamma>1$. Thus $W_E(L^p, \Lambda_\omega; 1, \gamma, p)$ does not depend on $\gamma>1$; this is also clear from (5.7), of course. From the last comment follows that

$$\|f\|_{W_E(L^p,\,\Lambda_{\omega^\alpha};\,1,\,\gamma,\,p)}\approx\|f\|_{W_E(L^p,\,\Lambda_{\omega^\alpha};\,1,\,\gamma^\alpha,\,p)}=\|f\|_{W_E(L^p,\,\Lambda_\omega;\,1,\,\gamma,\,p)}. \tag{5.8}$$

Let us prove that (5.4) and (5.2) imply (5.3):

THEOREM 5.2. *Let* $T: A_0+A_1\mapsto L^p+L^p_\omega$, *where* $0<p\leqslant\infty$, *be an* E*-quasilinear operator. Assume that for some* $\gamma>1$ *and all* $a\in A_0\cap A_1$

$$\sup_{-\infty<n<\infty}\left(\int_{\{\gamma^{n-1}<\omega\leqslant\gamma^n\}}|Ta|^p\,d\mu\right)^{1/p}\leqslant M_0\,\|a\|_{A_0} \tag{5.9}$$

and

$$\left(\int_{\Omega} |\omega Ta|^p \, d\mu\right)^{1/p} \leqslant M_1 \, \|a\|_{A_1}. \tag{5.10}$$

Then for all $0 < \theta < 1$ *we have*

$$\left(\int_{\Omega} |\omega^{\theta} Ta|^p \, d\mu\right)^{1/p} \leqslant C M_0^{1-\theta} M_1^{\theta} \, \|a\|_{(A_0, A_1)_{\theta, p; K}}.$$

Proof. Condition (5.9) can be written

$$Ta \in W_E(L^p, \Lambda_{\omega}; 1, 2, p)$$

which by (5.8) is equivalent to

$$Ta \in W_E(L^p, \Lambda_{\omega^2}; 1, 2, p).$$

By (5.6),

$$(\Lambda_{\omega^2}, L^p)_{1/2, p; E} = L^p_{\omega}$$

so that (5.10) can be written

$$\|Ta\|_{(\Lambda_{\omega^2}, L^p)_{\theta/2, p; E}} \leqslant M_1 \, \|a\|_{A_1}.$$

Thus, by Theorem 3.9 with $B_0 = L^p$ and $B_1 = \Lambda_{\omega^2}$ we have

$$\left(\int_{\Omega} |\omega^{\theta} Ta|^p \, d\mu\right)^{1/p} = \left(\frac{\theta p}{2}\right)^{1/p} \|Ta\|_{(\Lambda_{\omega^2}, L^p)_{\theta/2, p; E}}$$
$$\leqslant C M_0^{1-\theta} M_1^{\theta} \, \|a\|_{(A_0, A_1)_{\theta, p; K}}. \quad \blacksquare$$

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